it is unreasonable to expect that they can assimilate more than a small part of it, but they definitely do understand the distinction between real and ideal gases.

I found the preparation of the exercises very stimulating since I, like all students of my generation, never had the opportunity to examine equations of state of real gases very carefully. The type of information that now becomes readily available is illustrated in the figure; with the programs available, extension to other gases is a trivial matter.

Readers who would like copies of the three exercises are invited to write for them.

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# Computer Demonstrated Lenz-Ising Model

The Lenz-Ising Model has found wide use in the study of co-operative phenomena.[2] Gas-liquid and liquid-liquid critical phenomena, magnetic-curie points, and helix coil transitions are examples of the diversity of problems to which the model has been applied. Although the model is conceptually simple, its wonders are embedded deep within the mathematics used in its solution. The phenomena associated with the model in two dimensions (most spectacularly a critical temperature) are given *Deus ex machina* in most treatises on statistical mechanics leaving the student at an uncomfortable level of abstraction. This article describes a simple computer approximation to the Lenz-Ising model which demonstrates the predicted properties in the concrete terms of a computer printout of the ordering of an initially random Lenz-Ising lattice.

The heart of the program is two lattices, the Lenz-Ising lattice A, the elements of which can have the value $\pm 1$, and a second similar lattice B, used to store information. The lattice size and number of dimensions can be varied as can the energy $e$ needed to change the sign of an element as well as the temperature of the system.

Initially, lattice A is populated with random $\pm 1$ ($-1$ raised to a random integer power between 0 and 9). The individual elements of $\pm 1$ have two forces acting on them; random "thermal" force, and the force of the nearest neighbor interaction trying to make the sign of all the nearest neighbors the same.

The nearest neighbor force is determined for each element $a(i,j)$ of A with the help of an index $Q(i,j)$, the sum of all the nearest neighbor lattice elements; thus in two dimensions

$$Q(i,j) = a(i-1,j) + a(i+1,j) + \ldots + a(i,j+1)$$

If $Q(i,j)$ is less than 0, the majority of nearest neighbors are $-1$ and the element of lattice B, $b(i,j)$ is set equal to $-1$. If $Q(i,j)$ is greater than 0, $b(i,j)$ is set equal to $+1$; and if $Q(i,j) = 0$, $b(i,j) = a(i,j)$.

Having decided what the sign of an element would be from just nearest neighbor interaction, the program considers random thermal forces. To change the sign of an element the thermal energy $kT$ must overcome the energy of transition $E$ arising from an energy barrier $e$ as well as the force of its nearest neighbors. The energy of nearest neighbor interaction will be proportional to the absolute value of $Q$. The factor $\exp[-(e + |Q|/kT)]$ will give a number between 0 and 1 proportional to the probability that an element will have enough thermal energy to change sign. In order to determine whether this element is one of the lucky ones having enough energy, a random number between 0 and 1 is generated. If the random number is smaller than $\exp[-(e + |Q|/kT)]$, element $b(i,j)$ is changed to $-b(i,j)$.

After going through all of the elements of A and storing the information in B, the program sets lattice A equal to B and the process of determining which elements will change sign is begun over again.

In two dimensions, below the critical temperature, nearest neighbor forces succeed in ordering the lattice, while above the critical temperature no ordering occurs. The dynamics of the model, mimicking real critical phenomena, become very sluggish near the critical temperature. In fact, because of the great number of steps needed to order the system just below the critical temperature, this temperature is determined to better than one significant figure only with great patience.

Interestingly enough with $T = 0$ and therefore only nearest neighbor forces active, the lattice is seldom able to order itself. As soon as four or more adjacent elements form a rectangle they become impregnable to only nearest neighbor forces. At most an element of this configuration can have only two neighbors of different sign, and thus it cannot be forced to change its sign. It would seem that the random thermal forces are necessary to break up these otherwise unassailable domains.

The computer model was not intended to be rigorous, several better approximations to the Lenz-Ising lattice are evident. I think, however, that these improvements would be needless complications adding nothing to the pedagogic value of the program. The original computer program (in Basic) is available from the author on request.

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[1] Work performed at Dartmouth College.

[2] Brush, S. G., *Rev. Mod. Phys.*, 39, 883 (1967).